# A NOTE ON DEGENERATE AND ANAMOLOUS BOSONS

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**Abstract**

In this note it is shown that for a mono-energetic collection of Bosons, at a certain (non-zero) momentum or temperature, there is condensation while there is another momentum or temperature at which there is infinite dilution and below which the gas exhibits anamolous Fermionic behaviour.

We start with the well known formula for the occupation number in a Bose gas[1]:

$$\langle n_p \rangle = \frac{1}{z^{-1}e^{\beta\epsilon_p} - 1} \tag{1}$$

where

$$\beta \equiv 1/kT.$$

We also define,

$$z' \equiv \lambda^3/v \equiv zb, \lambda = \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$$

When $\epsilon_p = 0, z = 1$, we have from (1), $\langle n_0 \rangle = \infty$. This is the Bose-Einstein condensation.

Let us now consider a mono-energetic collection of Bosons:

$$\langle {n_p}' \rangle = \langle n_p \rangle \delta(p - p_0), \tag{2}$$

so that

$$kT \approx \epsilon_{p_0} = \frac{p_0^2}{2m}, \quad \lambda = (4\pi)^{1/2}\frac{\hbar}{p_0}, \quad \beta\epsilon_{p_0} \approx 1 \tag{3}$$

From (2) we get,

$$N = \frac{V}{\hbar^3}\int_0^\infty 4\pi p^2 dp\delta(p-p_0)\langle n_p\rangle, \tag{4}$$

where $N$ is the total number of particles and $V$ is the total volume. Using (1) and (3) in (4) we get,

$$1 = \frac{4\pi v p_0^2}{\hbar^3}\langle n_{p_0}\rangle = \left[\frac{(4\pi)^{5/2}}{p_0}\right]\frac{v}{\lambda^3}\left[\frac{1}{z^{-1}e-1}\right]$$

so that we have

$$z'^{-1}\left[\frac{(4\pi)^{5/2}}{p_0} - eb\right] = -1 \tag{5}$$

We can see from (5) that if

$$p_0 \approx \frac{(4\pi)^{5/2}}{eb-1} \tag{6}$$

(or at the corresponding temperature) then $z' \approx 1$. But this is the condition for condensation: Remembering that $\lambda$ is of the order of the particles' deBroglie wave length and $v$ is the average volume per particle, this means that the gas gets very densely packed. On the other hand, if

$$p_0 \approx \frac{(4\pi)^{5/2}}{eb} \tag{7}$$

then as can be seen from (5) $z' \approx 0$. In this case we have the opposite effect: The gas becomes very dilute. Finally, if

$$p_0 < \frac{(4\pi)^{5/2}}{eb}, \tag{8}$$

then equation (5) leads to a contradiction: We require that $z' < 0$, which is not possible.

The contradiction disappears if we realize that for momenta given by (8) or for the corresponding temperatures the Bosons effectively behave like

Fermions. In this case, the average occupation number is given, instead of (1), by,

$$\langle n_p \rangle = \frac{1}{z^{-1} e^{\beta \epsilon_p} + 1}$$

and instead of (5) we have,

$$z'^{-1} \left[ \frac{(4\pi)^{5/2}}{p_0} - eb \right] = +1$$

and condition(8) poses no problem. We now characterize $b$, which we have defined above in $z' = zb$. It is known that (cf.ref.[1]),

$$z' \left[ 1 - \frac{\langle n_0 \rangle}{N} \right] = g_{3/2}(z)$$

, where $\langle n_0 \rangle$ is the occupation number for energy or momentum $= 0$. As we are dealing with a mono-energetic Bose gas with non-zero energy (i.e. at non-zero temperature), we have,

$$\frac{\langle n_0 \rangle}{N} \approx 0,$$

so that,

$$z' \equiv zb \approx g_{3/2}(z) = z \left[ 1 + \frac{z}{2^{3/2}} + \frac{z^2}{3^{3/2}} + \ldots \right] \quad (0 \leq z \leq 1) \tag{9}$$

It easily follows that (cf.ref[1]),

$$1 < b < 2.6$$

. Infact if $z << 1$, we have from (9),

$$b \approx 1 \tag{10}$$

This is the case (7) where $z'$ and consequently $z$ are $\approx 0$. So (7) now becomes,

$$p_0 \approx \frac{(4\pi)^{5/2}}{e} \tag{11}$$

Coming now to the inequality (8), we observe that for Fermions, we have instead of (9), (cf.ref.[1]),

$$z' \equiv zb = f_{3/2}(z) = z - \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} \qquad (12)$$

If now the inequality (8) is nearly an equality, as we can choose, then, as in the case of (7), we will have, $z' \approx 0$ so that $z \approx 0$ also and from (12) we can see that $b$ satisfies (10). So the inequality (8) becomes,

$$p_0 < \frac{(4\pi)^{5/2}}{e} \qquad (13)$$

We finally come to the case of condensation given by (6). Here, from (9) it follows that

$$g_{3/2}(z) \approx 1,$$

which gives,

$$z \approx 0.7, b \approx 1.4.$$

Inserting this value of $b$ in (6), we get,

$$p_0 \approx \frac{(4\pi)^{5/2}}{1.4e-1} \qquad (14)$$

To sum up, for a collection of mono-energetic Bosons, (*i*) at the momentum given by (14) we have condensation; (*ii*) at the momentum given by (11) we have infinite dilution; (*iii*) for momenta given by (13) we have anamolous Fermionic behaviour.

# References

[1] K. Huang, "Statistical Mechanics", Wiley Eastern, New Delhi, 1975.